

# From uniform renewal theorem to uniform large and moderate deviations for renewal-reward processes

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## Abstract

A uniform key renewal theorem is deduced from the uniform Blackwell's renewal theorem. A uniform LDP (large deviations principle) for renewal-reward processes is obtained, and MDP (moderate deviations principle) is deduced under conditions much weaker than existence of exponential moments.

## Contents

1	Uniform renewal theorems	3
2	Uniform large deviations	8
3	Moderate deviations	12

## Introduction

An ordinary renewal-reward process  $S(\cdot)$  is a process of the form

$$S(t) = X_1 + \cdots + X_n \quad \text{for } \tau_1 + \cdots + \tau_n \leq t < \tau_1 + \cdots + \tau_{n+1};$$

here  $(\tau_1, X_1), (\tau_2, X_2), \dots$  are independent copies of a pair  $(\tau, X)$  of (generally, correlated) random variables such that  $\tau > 0$  a.s.

Large deviations principle (LDP) for  $S(t)$  (as  $t \rightarrow \infty$ ) is well-known when  $\tau$  and  $X$  have exponential moments. Otherwise the large deviations have peculiarity disclosed recently [4]. I prove moderate deviations principle (MDP) for  $S(t)$  requiring

- (1)  $\mathbb{E} \tau < \infty,$
- (2)  $\mathbb{E} \exp(\varepsilon X^2 - \tau) < \infty \quad \text{for some } \varepsilon > 0$

rather than  $\mathbb{E} \exp(\varepsilon|X|) < \infty$ ,  $\mathbb{E} \exp(\varepsilon\tau) < \infty$ . An example:  $X = \sqrt{\tau}$ ; MDP holds whenever  $\mathbb{E} \tau < \infty$ .

Conditions (1), (2) imply  $\mathbb{E} X^2 < \infty$  and are invariant under linear transformations of  $X$  and rescaling of  $\tau$  (see Remarks 3.1, 3.2), thus, we may restrict ourselves to the case

$$(3) \quad \mathbb{E} X = 0, \quad \mathbb{E} X^2 = 1, \quad \mathbb{E} \tau = 1.$$

**Theorem 1.** If (2), (3) are satisfied then

$$\lim_{x \rightarrow \infty, x/\sqrt{t} \rightarrow 0} \frac{1}{x^2} \ln \mathbb{P}(S(t) > x\sqrt{t}) = -\frac{1}{2}.$$

The limit in two variables  $t, x$  is taken; that is, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $t, x$  satisfying  $x > 1/\delta$ ,  $x/\sqrt{t} < \delta$  the function is  $\varepsilon$ -close to the limit.

Theorem 1 (MDP) will be deduced from Theorem 3, and Theorem 3 extends Theorem 2 (uniform LDP). The assumption for Theorem 2 is weaker than (2):

$$(4) \quad \forall \lambda \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \mathbb{E} \exp(\lambda X - \varepsilon \tau) < \infty.$$

(In combination with (1) it implies  $\mathbb{E} |X| < \infty$ , see Remark 2.5.)

**Theorem 2.** If (1), (4) hold and  $\mathbb{E} X = 0$  then for every  $\lambda$ , first, there exists one and only one  $\eta_\lambda \in [0, \infty)$  such that

$$(5) \quad \mathbb{E} \exp(\lambda X - \eta_\lambda \tau) = 1;$$

and second,

$$(6) \quad \frac{1}{t} \ln \mathbb{E} \exp \lambda S(t) = \eta_\lambda + O\left(\frac{1}{t}\right)$$

as  $t \rightarrow \infty$ , uniformly in  $\lambda \in [-C, -c] \cup [c, C]$  whenever  $0 < c < C < \infty$ .

**Theorem 3.** If (2) and (3) hold then

$$\eta_\lambda = \frac{1}{2} \lambda^2 + o(\lambda^2) \quad \text{as } \lambda \rightarrow 0,$$

and (6) holds uniformly in  $\lambda \in [-C, C]$  whenever  $0 < C < \infty$ .

# 1 Uniform renewal theorems

A uniform version of Blackwell's renewal theorem is available [5, Th. 1], [1, Th. 2.6(2), 2.7] and may be formulated as follows.

First, we define the *span* of a probability measure  $\mu$  on  $(0, \infty)$  as

$$\text{Span}(\mu) = \max(\{\delta > 0 : \mu(\{\delta, 2\delta, 3\delta, \dots\}) = 1\} \cup \{0\});$$

$\text{Span}(\mu) = 0$  if and only if  $\mu$  is nonlattice. A set  $M$  of probability measures on  $(0, \infty)$  will be called a *set of constant span*  $\delta$ , if  $\text{Span}(\mu) = \delta$  for all  $\mu \in M$ . Symbolically:  $\text{Span}(M) = \delta$ . Thus, a set of constant span 0 contains only nonlattice measures; a set of constant span  $\delta > 0$  contains only lattice measures of span  $\delta$  (rather than  $2\delta, 3\delta, \dots$ ).

Second, for every probability measure  $\mu$  on  $(0, \infty)$  we introduce the *renewal measure* as the sum of convolutions:

$$(1.1) \quad U_\mu = \sum_{n=0}^{\infty} \underbrace{\mu * \dots * \mu}_n$$

(the term for  $n = 0$  being the atom at the origin);  $U_\mu$  is not finite but locally finite, since  $\int e^{-t} U_\mu(dt) = \sum_n (\int e^{-t} \mu(dt))^n < \infty$ . It is well-known (see [2, p. 123]) that  $U_\mu((u, u+v]) \leq U_\mu([0, v])$  and moreover,

$$(1.2) \quad U_\mu((u, u+v]) \leq U_\mu([0, v]) \quad \text{for all } u, v \geq 0.$$

**1.3 Theorem.** ([5], [1]) Assume that a set  $M$  of probability measures on  $(0, \infty)$  is weakly compact (treated as a set of measures on  $\mathbb{R}$ ), is a set of constant span, and is uniformly integrable, that is,

$$\lim_{a \rightarrow +\infty} \sup_{\mu \in M} \int_{[a, \infty)} t \mu(dt) = 0.$$

Then in the nonlattice case ( $\text{Span}(M) = 0$ ), for every  $v > 0$ ,

$$U_\mu([u, u+v]) \rightarrow \frac{v}{\int t \mu(dt)} \quad \text{as } u \rightarrow \infty$$

uniformly in  $\mu \in M$ ; and in the lattice case ( $\text{Span}(M) = \delta$ )

$$U_\mu(\{n\delta\}) \rightarrow \frac{\delta}{\int t \mu(dt)} \quad \text{as } n \rightarrow \infty$$

uniformly in  $\mu \in M$ .

The uniform integrability of  $M$  ensures continuity of the function  $\mu \mapsto \int t \mu(dt)$  on  $M$ . We denote for convenience

$$\lambda_\mu = \frac{1}{\int t \mu(dt)};$$

by compactness,

$$(1.4) \quad 0 < \min_{\mu \in M} \lambda_\mu \leq \max_{\mu \in M} \lambda_\mu < \infty.$$

A uniform version of key renewal theorem follows. We start with the lattice case.

**1.5 Theorem.** Let  $M$  be a set of probability measures on  $(0, \infty)$  satisfying the conditions of Theorem 1.3,  $\text{Span}(M) = \delta > 0$ , and  $H$  a set of functions  $\{0, \delta, 2\delta, \dots\} \rightarrow \mathbb{R}$  such that

$$\sup_{h \in H} \sum_{k=0}^{\infty} |h(k\delta)| < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{h \in H} \sum_{k=n}^{\infty} |h(k\delta)| = 0.$$

Then

$$(U_\mu * h)(n\delta) \rightarrow \delta \lambda_\mu \sum_{k=0}^{\infty} h(k\delta) \quad \text{as } n \rightarrow \infty$$

uniformly in  $\mu \in M$  and  $h \in H$ .

*Proof.* By (1.2),  $U_\mu(\{n\delta\}) \leq U_\mu(\{0\}) = 1$  for all  $\mu$  and  $n$ . By Theorem 1.3,  $U_\mu(\{n\delta\}) \rightarrow \delta \lambda_\mu$  as  $n \rightarrow \infty$ , uniformly in  $\mu \in M$ . Lemma 1.6 (below) completes the proof.  $\square$

**1.6 Lemma.** Let  $U$  and  $H$  be sets of functions  $\{0, 1, 2, \dots\} \rightarrow \mathbb{R}$  such that

$$\sup_{u \in U} \sup_n |u(n)| < \infty,$$

the limit  $u(\infty) = \lim_{n \rightarrow \infty} u(n)$  exists uniformly in  $u \in U$ ;

$$\sup_{h \in H} \sum_n |h(n)| < \infty;$$

$$\sum_{n=N}^{\infty} |h(n)| \rightarrow 0 \text{ as } N \rightarrow \infty, \text{ uniformly in } h \in H.$$

Then

$$(u * h)(n) \rightarrow u(\infty) \sum_{k=0}^{\infty} h(k) \text{ as } n \rightarrow \infty, \text{ uniformly in } u \in U \text{ and } h \in H.$$

*Proof.* Denoting  $\|u\|_\infty = \sup_n |u(n)|$ ,  $\|h\|_1 = \sum_n |h(n)|$  and  $\Sigma(h) = \sum_n h(n)$  we have  $\|u * h\|_\infty \leq \|u\|_\infty \|h\|_1$ ,  $|u(\infty)| \leq \|u\|_\infty$  and  $|\Sigma(h)| \leq \|h\|_1$ . For arbitrary  $N \in \{0, 1, 2, \dots\}$  and  $h \in H$  we introduce  $h_N, h^N : \{0, 1, 2, \dots\} \rightarrow \mathbb{R}$  by  $h_N(n) = h(n)$  for  $n \leq N$ ,  $h_N(n) = 0$  for  $n > N$ , and  $h^N = h - h_N$ . We have  $\sup_{u \in U} \|u\|_\infty < \infty$ ,  $\sup_{h \in H} \|h\|_1 < \infty$ , and  $\sup_{h \in H} \|h^N\|_1 \rightarrow 0$  as  $N \rightarrow \infty$ . For arbitrary  $N$  and all  $n \geq N$ ,

$$\begin{aligned} |(u * h)(n) - u(\infty)\Sigma(h)| &\leq \\ &\leq |(u * h_N)(n) - u(\infty)\Sigma(h_N)| + |(u * h^N)(n) - u(\infty)\Sigma(h^N)| \leq \\ &\leq \left| \sum_{k=0}^N u(n-k)h(k) - u(\infty) \sum_{k=0}^N h(k) \right| + |(u * h^N)(n)| + |u(\infty)\Sigma(h^N)| \leq \\ &\leq \sum_{k=0}^N |u(n-k) - u(\infty)| |h(k)| + \|u * h^N\|_\infty + |u(\infty)| |\Sigma(h^N)| \leq \\ &\leq \|h\|_1 \sup_{k \geq n-N} |u(k) - u(\infty)| + 2\|u\|_\infty \|h^N\|_1; \end{aligned}$$

given  $\varepsilon > 0$ , we choose  $N$  such that  $\|u\|_\infty \|h^N\|_1 \leq \varepsilon$  for all  $u \in U$  and  $h \in H$ ; then for all  $n$  large enough we have  $\|h\|_1 \sup_{k \geq n-N} |u(k) - u(\infty)| \leq \varepsilon$  for all  $u \in U$  and  $h \in H$ , and finally,  $|(u * h)(n) - u(\infty)\Sigma(h)| \leq 3\varepsilon$ .  $\square$

The nonlattice case needs more effort. Recall that a function  $h : [0, \infty) \rightarrow \mathbb{R}$  is called *directly Riemann integrable*, if two limits exist and are equal (and finite):

$$\lim_{\delta \rightarrow 0+} \delta \sum_{n=0}^{\infty} \inf_{[n\delta, n\delta+\delta)} h(\cdot) = \lim_{\delta \rightarrow 0+} \delta \sum_{n=0}^{\infty} \sup_{[n\delta, n\delta+\delta)} h(\cdot).$$

**1.7 Definition.** A set  $H$  of functions  $[0, \infty) \rightarrow \mathbb{R}$  is *uniformly directly Riemann integrable*, if

$$\begin{aligned} \sup_{h \in H} \sum_{n=0}^{\infty} \sup_{[n, n+1)} |h(\cdot)| &< \infty, \\ \sum_{n=N}^{\infty} \sup_{[n, n+1)} |h(\cdot)| &\rightarrow 0 \quad \text{as } N \rightarrow \infty, \text{ uniformly in } h \in H; \\ \delta \sum_{n=0}^{\infty} \left( \sup_{[n\delta, n\delta+\delta)} h(\cdot) - \inf_{[n\delta, n\delta+\delta)} h(\cdot) \right) &\rightarrow 0 \quad \text{as } \delta \rightarrow 0+, \text{ uniformly in } h \in H. \end{aligned}$$

**1.8 Remark.** If  $\sup_{h \in H} \sum_{n=0}^{\infty} \sup_{[n\delta, n\delta+\delta)} |h(\cdot)| < \infty$  for some  $\delta$  then it holds for all  $\delta$ . Proof: given  $\delta_1, \delta_2 > 0$ , we consider  $A = \{(n_1, n_2) : [n_1\delta_1, n_1\delta_1 +$

$\delta_1) \cap [n_2\delta_2, n_2\delta_2 + \delta_2) \neq \emptyset\}$ , note that  $\#\{n_1 : (n_1, n_2) \in A\} \leq \frac{\delta_2}{\delta_1} + 2$ , and get

$$\begin{aligned} \sum_{n_1=0}^{\infty} \sup_{[n_1\delta_1, n_1\delta_1+\delta_1)} |h(\cdot)| &\leq \sum_{n_1=0}^{\infty} \max_{n_2: (n_1, n_2) \in A} \sup_{[n_2\delta_2, n_2\delta_2+\delta_2)} |h(\cdot)| \leq \\ &\leq \sum_{(n_1, n_2) \in A} \sup_{[n_2\delta_2, n_2\delta_2+\delta_2)} |h(\cdot)| \leq \left(\frac{\delta_2}{\delta_1} + 2\right) \sum_{n_2=0}^{\infty} \sup_{[n_2\delta_2, n_2\delta_2+\delta_2)} |h(\cdot)|. \end{aligned}$$

Thus, the first two conditions of Def. 1.7 may be reformulated as

$$\begin{aligned} \sup_{h \in H} \sum_{n=0}^{\infty} \sup_{[n\delta, n\delta+\delta)} |h(\cdot)| &< \infty, \\ \sum_{n=N}^{\infty} \sup_{[n\delta, n\delta+\delta)} |h(\cdot)| &\rightarrow 0 \quad \text{as } N \rightarrow \infty, \text{ uniformly in } h \in H \end{aligned}$$

for some (therefore, all)  $\delta > 0$ .

**1.9 Remark.** Similarly,

$$\delta \sum_{n: n\delta > N} \left( \sup_{[n\delta, n\delta+\delta)} h(\cdot) - \inf_{[n\delta, n\delta+\delta)} h(\cdot) \right) \leq (1 + 2\delta) \sum_{n=N}^{\infty} \sup_{[n, n+1)} |h(\cdot)|.$$

Thus, the third condition of Def. 1.7 may be reformulated as uniform Riemann integrability on bounded intervals: for every  $N$ ,

$$\delta \sum_{n \geq 0: n\delta \leq N} \left( \sup_{[n\delta, n\delta+\delta)} h(\cdot) - \inf_{[n\delta, n\delta+\delta)} h(\cdot) \right) \rightarrow 0 \quad \text{as } \delta \rightarrow 0+, \text{ uniformly in } h \in H.$$

**1.10 Remark.** If each  $h \in H$  is a decreasing function  $[0, \infty) \rightarrow [0, \infty)$  then  $H$  is uniformly directly Riemann integrable if and only if

$$\begin{aligned} \sup_{h \in H} h(0) &< \infty, \quad \sup_{h \in H} \int_0^{\infty} h(s) ds < \infty, \quad \text{and} \\ \sup_{h \in H} \int_a^{\infty} h(s) ds &\rightarrow 0 \quad \text{as } a \rightarrow \infty. \end{aligned}$$

By taking differences, a similar result can be obtained for functions of uniformly bounded variation on  $[0, \infty)$  (rather than decreasing).

**1.11 Theorem.** Let  $M$  be a set of probability measures on  $(0, \infty)$  satisfying the conditions of Theorem 1.3,  $\text{Span}(M) = 0$ , and  $H$  a uniformly directly Riemann integrable set of functions  $[0, \infty) \rightarrow \mathbb{R}$ . Then

$$(U_{\mu} * h)(t) \rightarrow \lambda_{\mu} \int_0^{\infty} h(s) ds \quad \text{as } t \rightarrow \infty$$

uniformly in  $\mu \in M$  and  $h \in H$ .

Here is a generalization of Lemma 1.6, to be used in the proof of the theorem.

**1.12 Lemma.** Let  $H$  be as in Lemma 1.6, and  $V$  a set of functions  $\{0, 1, 2, \dots\} \times [0, \infty) \rightarrow \mathbb{R}$  such that, first,  $\sup_{v \in V} \sup_n \sup_t |v_n(t)| < \infty$ , and second, the limit  $v(\infty) = \lim_{t \rightarrow \infty} v_n(t)$  exists uniformly in  $v \in V$  for every  $n$ , and does not depend on  $n$ . Then

$$\sum_{n=0}^{\infty} h(n) v_n(t) \rightarrow v(\infty) \sum_{n=0}^{\infty} h(n) \text{ as } t \rightarrow \infty, \text{ uniformly in } v \in V \text{ and } h \in H.$$

*Proof.* The proof of Lemma 1.6 needs only trivial modifications:  $\sum_n h(n) v_n(t)$  instead of  $(u * h)(n)$ ;  $\sum_{n=0}^N |v_n(t) - v(\infty)| |h(n)|$  instead of  $\sum_{k=0}^N |u(n - k) - u(\infty)| |h(k)|$ ; and  $\max_{n=0, \dots, N} |v_n(t) - v(\infty)|$  (for large  $t$ ) instead of  $\sup_{k \geq n-N} |u(k) - u(\infty)|$  (for large  $n$ ). Also,  $\|v\|_{\infty} = \sup_{n,t} |v_n(t)|$ .  $\square$

Here is a special case of Theorem 1.11 for step functions.

**1.13 Lemma.** Assume that  $M$  and  $H$  are as in Theorem 1.11,  $\delta > 0$ , and every  $h \in H$  is constant on each  $[n\delta, n\delta + \delta)$ . Then the conclusion of Theorem 1.11 holds.

*Proof.* Lemma 1.12 will be applied to  $\tilde{H}$  and  $V$ , where  $\tilde{H}$  consists of all  $\tilde{h}$  of the form  $\tilde{h}(n) = h(n\delta)$  for  $h \in H$ , and  $V$  consists of all  $v$  of the form

$$v_n(\cdot) = U_{\mu} * \mathbb{1}_{[n\delta, n\delta + \delta)}$$

for  $\mu \in M$ ; that is,  $v_n(t) = U_{\mu}((t - n\delta - \delta, t - n\delta])$ . By (1.2),

$$v_n(t) \leq U_{\mu}([0, \delta)) \leq e^{\delta} \int e^{-s} U_{\mu}(ds) = \frac{e^{\delta}}{1 - \int e^{-s} \mu(ds)};$$

by compactness of  $M$ ,

$$\sup_{v,n,t} |v_n(t)| \leq \frac{e^{\delta}}{1 - \max_{\mu} \int e^{-s} \mu(ds)} < \infty.$$

By Theorem 1.3, for every  $n$ ,  $v_n(t) \rightarrow \lambda_{\mu} \delta$  as  $t \rightarrow \infty$ , uniformly in  $v$ . Thus,  $V$  satisfies the conditions of Lemma 1.12. By Remark 1.8,  $\tilde{H}$  satisfies the conditions (for  $H$ ) of Lemma 1.12, that is, of Lemma 1.6. It remains to apply Lemma 1.12 and take into account that  $v(\infty) = \lambda_{\mu} \delta$ ,  $\delta \sum_n \tilde{h}(n) = \int_0^{\infty} h(s) ds$  and  $\sum_n \tilde{h}(n) v_n(\cdot) = U_{\mu} * h$  since  $\sum_n h(n\delta) \mathbb{1}_{[n\delta, n\delta + \delta)} = h$ .  $\square$

*Proof of Theorem 1.11.* For arbitrary  $\delta > 0$  and  $h \in H$  we introduce  $h_\delta^-, h_\delta^+ : [0, \infty) \rightarrow \mathbb{R}$  by

$$h_\delta^-(t) = \inf_{[n\delta, n\delta+\delta)} h(\cdot), \quad h_\delta^+(t) = \sup_{[n\delta, n\delta+\delta)} h(\cdot) \quad \text{for } t \in [n\delta, n\delta + \delta),$$

then  $h_\delta^- \leq h \leq h_\delta^+$ . The sets  $H_\delta^- = \{h_\delta^- : h \in H\}$ ,  $H_\delta^+ = \{h_\delta^+ : h \in H\}$  are uniformly directly Riemann integrable by the arguments of Remarks 1.8, 1.9. Applying Lemma 1.13 to  $M$  and  $H_\delta^\pm$  we get

$$(U_\mu * h_\delta^\pm)(t) \rightarrow \lambda_\mu \int_0^\infty h_\delta^\pm(s) ds \quad \text{as } t \rightarrow \infty$$

uniformly in  $\mu \in M$  and  $h \in H$ .

Given  $\varepsilon > 0$ , we choose  $\delta = \delta_\varepsilon$  such that  $\int |h_\delta^\pm(t) - h(t)| dt \leq \varepsilon$  for all  $h \in H$ . Then we choose  $t_\varepsilon$  such that for all  $t \geq t_\varepsilon$ ,  $\mu \in M$  and  $h \in H$ ,

$$\left| (U_\mu * h_\delta^\pm)(t) - \lambda_\mu \int_0^\infty h_\delta^\pm(s) ds \right| \leq \varepsilon.$$

We get

$$\begin{aligned} (U_\mu * h)(t) - \lambda_\mu \int h(s) ds &\leq \\ &\leq (U_\mu * h_\delta^+)(t) - \lambda_\mu \int_0^\infty h_\delta^+(s) ds + \lambda_\mu \left( \int_0^\infty h_\delta^+(s) ds - \int_0^\infty h(s) ds \right) \leq \\ &\leq \varepsilon + \lambda_\mu \varepsilon \end{aligned}$$

and a similar lower bound; thus, using (1.4),

$$\left| (U_\mu * h)(t) - \lambda_\mu \int h(s) ds \right| \leq \varepsilon \left( 1 + \max_{\mu \in M} \lambda_\mu \right)$$

for all  $t \geq t_\varepsilon$ ,  $\mu \in M$  and  $h \in H$ . □

## 2 Uniform large deviations

Theorem 2 is proved in this section.

Exponential moments of a renewal-reward process boil down to a renewal equation, see [3, Th. 5], and therefore to an auxiliary renewal process, as sketched below.

Having  $\eta_\lambda$  satisfying (5) for a given  $\lambda$ , we introduce a random variable  $\tau_\lambda$  distributed so that

$$(2.1) \quad \mathbb{E} f(\tau_\lambda) = \mathbb{E} (e^{\lambda X - \eta_\lambda \tau} f(\tau))$$



for all bounded Borel functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . That is,

$$(2.2) \quad \begin{aligned} &\tau \text{ is distributed } \mu, \quad \tau_\lambda \text{ is distributed } \mu_\lambda, \\ &\frac{d\mu_\lambda}{d\mu}(\tau) = \mathbb{E}(e^{\lambda X - \eta_\lambda \tau} | \tau). \end{aligned}$$

Then, using the notation  $\mathbb{E}(Z; A) = \mathbb{E}(Z \cdot \mathbb{1}_A)$ , we have

$$\mathbb{E}(e^{\lambda S(t)}; \tau_1 + \dots + \tau_n \leq t < \tau_1 + \dots + \tau_{n+1}) = e^{\eta_\lambda t} \mathbb{E} h_\lambda(t - \tau_{\lambda,1} - \dots - \tau_{\lambda,n}),$$

where  $\tau_{\lambda,1}, \tau_{\lambda,2}, \dots$  are independent copies of  $\tau_\lambda$ , and

$$(2.3) \quad h_\lambda(t) = \begin{cases} e^{-\eta_\lambda t} \mathbb{P}(\tau_\lambda > t) & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

Summing up we get

$$(2.4) \quad \mathbb{E} e^{\lambda S(t)} = e^{\eta_\lambda t} \sum_{n=0}^{\infty} \mathbb{E} h_\lambda(t - \tau_{\lambda,1} - \dots - \tau_{\lambda,n}) = e^{\eta_\lambda t} (U_\lambda * h_\lambda)(t);$$

here  $U_\lambda = U_{\mu_\lambda}$  is the renewal measure (recall (1.1)).

Recall assumptions (1)  $\mathbb{E} \tau < \infty$  and (4)  $\forall \lambda \in \mathbb{R} \forall \varepsilon > 0 \quad \mathbb{E} \exp(\lambda X - \varepsilon \tau) < \infty$ .

**2.5 Remark.** Assumptions (1), (4) imply  $\mathbb{E} |X| < \infty$ .

*Proof.*  $|X| \leq \tau + e^{|X|-\tau} \leq \tau + e^{-X-\tau} + e^{X-\tau}$  is integrable.  $\square$

From now on, till the end of this section, we assume the conditions of Theorem 2; that is, (1), (4), and  $\mathbb{E} X = 0$ . We also assume that  $\mathbb{P}(X = 0) \neq 1$ ; otherwise Theorem 2 is trivial.

**2.6 Lemma.** Maps  $(\lambda, \eta) \mapsto \exp(\lambda X - \eta \tau)$  and  $(\lambda, \eta) \mapsto \tau \exp(\lambda X - \eta \tau)$  are continuous from  $\mathbb{R} \times (0, \infty)$  to the space  $L_1$  of integrable random variables.

*Proof.* It is sufficient to prove the continuity on  $[-C, C] \times [2\varepsilon, \infty)$  for arbitrary  $C, \varepsilon > 0$ . Also, it is sufficient to consider the map  $(\lambda, \eta) \mapsto e^{\varepsilon \tau} \exp(\lambda X - \eta \tau)$ , since  $\tau \leq \frac{1}{\varepsilon} e^{\varepsilon \tau}$  a.s. We apply the dominated convergence theorem, taking into account that  $\exp(-CX - \varepsilon \tau) + \exp(CX - \varepsilon \tau)$  is an integrable majorant of  $e^{\varepsilon \tau} \exp(\lambda X - \eta \tau)$  for all  $\lambda \in [-C, C]$  and  $\eta \in [2\varepsilon, \infty)$ .  $\square$

**2.7 Lemma.** For every  $\lambda$  there is one and only one  $\eta_\lambda$  satisfying (5)  $\mathbb{E} \exp(\lambda X - \eta_\lambda \tau) = 1$ , and the function  $\lambda \mapsto \eta_\lambda$  is continuous on  $\mathbb{R}$ .

*Proof.* The function  $\psi : \mathbb{R} \times (0, \infty) \rightarrow (0, \infty)$  defined by  $\psi(\lambda, \eta) = \mathbb{E} \exp(\lambda X - \eta \tau)$  is continuous by Lemma 2.6. For every  $\lambda$  the function  $\psi(\lambda, \cdot)$  is strictly decreasing,  $\psi(\lambda, +\infty) = 0$ , and (possibly, infinite)  $\psi(\lambda, 0+) = \mathbb{E} \exp \lambda X > \exp \lambda \mathbb{E} X = 1$  provided that  $\lambda \neq 0$ . Thus, for  $\lambda \neq 0$  we get unique  $\eta_\lambda > 0$ ; and trivially,  $\eta_0 = 0$ .

It remains to prove continuity of the function  $\lambda \mapsto \eta_\lambda$ . Given  $\lambda_0 \neq 0$  and  $\varepsilon < \eta_{\lambda_0}$  we note that  $\psi(\lambda_0, \eta_{\lambda_0} + \varepsilon) < 1 = \psi(\lambda_0, \eta_{\lambda_0}) < \psi(\lambda_0, \eta_{\lambda_0} - \varepsilon)$  and take  $\delta > 0$  such that  $\psi(\lambda, \eta_{\lambda_0} + \varepsilon) < 1 = \psi(\lambda, \eta_\lambda) < \psi(\lambda, \eta_{\lambda_0} - \varepsilon)$  and therefore  $\eta_{\lambda_0} - \varepsilon < \eta_\lambda < \eta_{\lambda_0} + \varepsilon$  for all  $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ . For  $\lambda_0 = 0$  we use a one-sided version of the same argument: given  $\varepsilon > 0$ , we take  $\delta > 0$  such that  $\psi(\lambda, \varepsilon) < 1$  and therefore  $\eta_\lambda < \varepsilon$  for all  $\lambda \in (-\delta, \delta)$ .  $\square$

Recall measures  $\mu, \mu_\lambda$  given by (2.2).

**2.8 Lemma.** The function  $\lambda \mapsto \mu_\lambda$  is continuous from  $(-\infty, 0) \cup (0, \infty)$  to the space of measures with the norm topology.

*Proof.* We have  $d\mu_\lambda/d\mu = \varphi_\lambda$ , where  $\varphi_\lambda$  is defined by  $\varphi_\lambda(\tau) = \mathbb{E}(e^{\lambda X - \eta_\lambda \tau} | \tau)$ . If  $\lambda_n \rightarrow \lambda \neq 0$  then, using Lemmas 2.6 and 2.7,  $\|\mu_{\lambda_n} - \mu_\lambda\| = \int |\varphi_{\lambda_n} - \varphi_\lambda| d\mu = \mathbb{E} |\varphi_{\lambda_n}(\tau) - \varphi_\lambda(\tau)| = \mathbb{E} |\mathbb{E}(e^{\lambda_n X - \eta_{\lambda_n} \tau} | \tau) - \mathbb{E}(e^{\lambda X - \eta_\lambda \tau} | \tau)| \leq \mathbb{E} |e^{\lambda_n X - \eta_{\lambda_n} \tau} - e^{\lambda X - \eta_\lambda \tau}| \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**2.9 Lemma.** The set  $\{\mu_\lambda : \lambda \in [-C, -c] \cup [c, C]\}$  satisfies the conditions of Theorem 1.3 whenever  $0 < c < C < \infty$ .

*Proof.* Lemma 2.8 ensures compactness (even in a topology stronger than needed). For every  $\lambda$  measures  $\mu_\lambda$  and  $\mu$  are mutually absolutely continuous, therefore  $\text{Span}(\mu_\lambda) = \text{Span}(\mu)$ . It remains to prove uniform integrability. Using (2.1) we have  $\int_{[a, \infty)} t \mu_\lambda(dt) = \mathbb{E}(\tau_\lambda \mathbb{1}_{[a, \infty)}(\tau_\lambda)) = \mathbb{E}(\tau \exp(\lambda X - \eta_\lambda \tau) \mathbb{1}_{[a, \infty)}(\tau)) \rightarrow 0$  as  $a \rightarrow \infty$  uniformly in  $\lambda \in [-C, -c] \cup [c, C]$ , since random variables  $\tau \exp(\lambda X - \eta_\lambda \tau)$  for these  $\lambda$  are a compact subset of  $L_1$  by Lemmas 2.6, 2.7.  $\square$

Recall functions  $h_\lambda$  given by (2.3).

**2.10 Lemma.** The set  $\{h_\lambda : \lambda \in [-C, -c] \cup [c, C]\}$  is uniformly directly Riemann integrable whenever  $0 < c < C < \infty$ .

*Proof.* Follows from Remark 1.10, (1.4) and the uniform integrability of measures  $\mu_\lambda$ , since  $h_\lambda(0) \leq 1$  and

$$\begin{aligned} \int_a^\infty h_\lambda(t) dt &= \int_a^\infty e^{-\eta_\lambda t} \mathbb{P}(\tau_\lambda > t) dt \leq \int_a^\infty \mu_\lambda((t, \infty)) dt = \\ &= \int_a^\infty (t - a) \mu_\lambda(dt) \leq \int_{[a, \infty)} t \mu_\lambda(dt). \end{aligned}$$

□

**2.11 Lemma.** The map  $\lambda \mapsto h_\lambda$  is continuous from  $(-\infty, 0) \cup (0, \infty)$  to  $L_1(0, \infty)$ .

*Proof.* For every  $t > 0$ ,  $h_\lambda(t) = e^{-\eta_\lambda t} \mu_\lambda((t, \infty))$  is continuous in  $\lambda \neq 0$  by Lemmas 2.7, 2.8. Also,  $h_\lambda(t) \leq 1$ . Thus,  $\lambda \mapsto h_\lambda|_{(0,a)} \in L_1(0,a)$  is continuous (by the dominated convergence theorem). The limit as  $a \rightarrow \infty$  is locally uniform around a given  $\lambda \neq 0$  due to the uniform integrability of  $h_\lambda$  (proved in Lemma 2.10). □

Functions  $h_\lambda$  as elements of  $L_1(0, \infty)$  are relevant in the nonlattice case, when  $\text{Span}(\mu) = 0$ , while in the lattice case, when  $\text{Span}(\mu) = \delta > 0$ , we treat sequences  $(h_\lambda(k\delta))_{k=0}^\infty$  as elements of the space  $l_1$  of summable sequences.

**2.12 Lemma.** Let  $\text{Span}(\mu) = \delta > 0$ , then the map  $\lambda \mapsto (h_\lambda(k\delta))_{k=0}^\infty$  is continuous from  $(-\infty, 0) \cup (0, \infty)$  to  $l_1$ .

*Proof.* We have

$$h_\lambda(k\delta) = \frac{1}{\delta} \int_{k\delta}^{k\delta+\delta} e^{\eta_\lambda(t-k\delta)} h_\lambda(t) dt,$$

since the function  $t \mapsto e^{\eta_\lambda t} h_\lambda(t) = \mu_\lambda((t, \infty))$  is constant on  $[k\delta, k\delta + \delta)$ . We apply Lemma 2.11, taking into account continuity of  $\lambda \mapsto \eta_\lambda$ . □

*Proof of Theorem 2.* Existence and uniqueness of  $\eta_\lambda$  satisfying (5) are ensured by Lemma 2.7.

We reformulate (6) as existence of  $T \in (0, \infty)$  such that

$$(2.13) \quad \sup_{\lambda \in [-C, -c] \cup [c, C], t \in [T, \infty)} \left| -\eta_\lambda t + \ln \mathbb{E} e^{\lambda S(t)} \right| < \infty.$$

The set  $M = \{\mu_\lambda : \lambda \in [-C, -c] \cup [c, C]\}$  satisfies the conditions of Theorem 1.3 by Lemma 2.9.

By 1.4,  $\int t \mu_\lambda(dt)$  is bounded away from 0 and  $\infty$  for  $\lambda \in [-C, -c] \cup [c, C]$ . The rest of the proof of (2.13) splits in two cases.

Nonlattice case:  $\text{Span}(\mu) = 0$ .

The set  $H = \{h_\lambda : \lambda \in [-C, -c] \cup [c, C]\}$  is uniformly directly Riemann integrable by Lemma 2.10. By (2.4) and Theorem 1.11,

$$e^{-\eta_\lambda t} \mathbb{E} e^{\lambda S(t)} \rightarrow \frac{\int_0^\infty h_\lambda(s) ds}{\int s \mu_\lambda(ds)} \quad \text{as } t \rightarrow \infty$$

uniformly in  $\lambda \in [-C, -c] \cup [c, C]$ . In order to get (2.13) it remains to check that the right-hand side is bounded away from 0 and  $\infty$ . For the

denominator, see above. For the numerator, use continuity of the function  $\lambda \mapsto \int_0^\infty h(t) dt$  for  $\lambda \neq 0$  (Lemma 2.11).

Lattice case:  $\text{Span}(\mu) = \delta > 0$ .

The set  $H$  of restrictions to  $\{0, \delta, 2\delta, \dots\}$  of the functions  $h_\lambda$  for  $\lambda \in [-C, -c] \cup [c, C]$  satisfies the conditions of Theorem 1.5 by Lemma 2.12 (via compactness in  $l_1$ ). By (2.4) and Theorem 1.5,

$$e^{-\eta_\lambda n\delta} \mathbb{E} e^{\lambda S(n\delta)} \rightarrow \frac{\delta \sum_{k=0}^\infty h_\lambda(k\delta)}{\int s \mu_\lambda(ds)} \quad \text{as } n \rightarrow \infty$$

uniformly in  $\lambda \in [-C, -c] \cup [c, C]$ . The function  $\lambda \mapsto \sum_{k=0}^\infty h_\lambda(k\delta)$  is continuous for  $\lambda \neq 0$  by Lemma 2.12, therefore the sum is bounded away from 0 and  $\infty$  for  $\lambda \in [-C, -c] \cup [c, C]$  (it cannot vanish since  $h_\lambda(0) = 1$ ), which leads to (2.13). Thus we get (2.13) for  $t$  running on the lattice, which is enough, since  $S(\cdot)$  is constant on  $[k\delta, k\delta + \delta)$  (and  $\eta_\lambda$  is bounded).  $\square$

### 3 Moderate deviations

Theorems 1 and 3 are proved in this section.

In order to use small  $\lambda$  we need (2):  $\mathbb{E} \exp(\varepsilon X^2 - \tau) < \infty$  for some  $\varepsilon > 0$ .

**3.1 Remark.** Assumptions (1), (2) imply  $\mathbb{E} X^2 < \infty$ .

*Proof.*  $\varepsilon X^2 \leq \tau + e^{\varepsilon X^2 - \tau}$  is integrable.  $\square$

**3.2 Remark.** Assumption (2) is invariant under linear transformations of  $X$ , and rescaling of  $\tau$ ; also, (2) implies (4).

*Proof.* Rescaling  $X$ :  $\mathbb{E} \exp((c^{-2}\varepsilon)(cX)^2 - \tau) = \mathbb{E} \exp(\varepsilon X^2 - \tau) < \infty$ .

Shifting  $X$ :  $\mathbb{E} \exp(\frac{\varepsilon}{2}(X+c)^2 - \tau) \leq \mathbb{E} \exp(\frac{\varepsilon}{2}(X-c)^2 + \frac{\varepsilon}{2}(X+c)^2 - \tau) = e^{c^2\varepsilon} \mathbb{E} \exp(\varepsilon X^2 - \tau) < \infty$ .

Rescaling  $\tau$ :  $\mathbb{E} \exp(c\varepsilon X^2 - c\tau) = \mathbb{E} (\exp(\varepsilon X^2 - \tau))^c \leq (\mathbb{E} \exp(\varepsilon X^2 - \tau))^c < \infty$  for  $c \in (0, 1)$ , and  $\mathbb{E} \exp(\varepsilon X^2 - c\tau) \leq \mathbb{E} \exp(\varepsilon X^2 - \tau) < \infty$  for  $c \in [1, \infty)$ .

Finally, (2) implies (4) since  $\mathbb{E} \exp(\delta X^2 - \tau) < \infty$  implies  $\mathbb{E} \exp(\varepsilon \delta X^2 - \varepsilon \tau) < \infty$  (assuming  $0 < \varepsilon < 1$ ) and therefore  $\mathbb{E} \exp(\lambda X - \varepsilon \tau) \leq \mathbb{E} \exp(\frac{\lambda^2}{4\varepsilon\delta} + \varepsilon \delta X^2 - \varepsilon \tau) < \infty$ .  $\square$

From now on, till the end of this section, we assume the conditions of Theorem 3; that is, (2), and (3):  $\mathbb{E} X = 0$ ,  $\mathbb{E} X^2 = 1$ ,  $\mathbb{E} \tau = 1$ . Conditions of Theorem 2 follow, since (2) implies (4) by Remark 3.2.

Here is an analytic fact that will give us some integrable majorants.

**3.3 Lemma.** For all  $a, \varepsilon, \Lambda \in (0, \infty)$ ,

$$\sup_{t>0, x>0, \lambda \in (0, \Lambda)} \frac{(1+t+x^2) \exp(\lambda x - a\lambda^2 t)}{t + \exp(\varepsilon x^2 - t)} < \infty.$$

*Proof.* Denoting this supremum by  $S(a, \varepsilon, \Lambda)$  we observe that  $S(a, \varepsilon, \Lambda) \leq \max(1, c^2) S(c^{-2}a, c^2\varepsilon, c\Lambda)$  for arbitrary  $c > 0$  (by rescaling,  $x \mapsto cx$  and  $\lambda \mapsto c^{-1}\lambda$ ). Thus, we restrict ourselves to  $\varepsilon = 1$ .

We note that

$$\max_{\lambda \in \mathbb{R}} (\lambda x - a\lambda^2 t) = \frac{x^2}{4at}.$$

We choose  $\alpha, \beta > 0$  such that  $\alpha > 1$  and  $\beta^2 < 4a(\alpha^2 - 1)$  (for instance,  $\alpha = 2$  and  $\beta = 3\sqrt{a}$ ) and consider three cases.

Case 1:  $x \leq \alpha\sqrt{t}$ .

We note that  $t + \exp(x^2 - t) \geq t + e^{-t} \geq \max(t, 1)$ , thus,

$$\begin{aligned} \frac{(1+t+x^2) \exp(\lambda x - a\lambda^2 t)}{t + \exp(x^2 - t)} &\leq \frac{(1+t+x^2) \exp \frac{x^2}{4at}}{\max(t, 1)} \leq \\ &\leq \frac{(1+t+\alpha^2 t) \exp \frac{\alpha^2}{4a}}{\max(t, 1)} \leq (2+\alpha^2) \exp \frac{\alpha^2}{4a}. \end{aligned}$$

Case 2:  $\alpha\sqrt{t} \leq x \leq \beta t$ .

$$\begin{aligned} \frac{(1+t+x^2) \exp(\lambda x - a\lambda^2 t)}{t + \exp(x^2 - t)} &\leq \frac{(1+t+x^2) \exp \frac{x^2}{4at}}{\exp(\alpha^2 t - t)} \leq \\ &\leq (1+t+\beta^2 t^2) \exp \left( \frac{\beta^2 t^2}{4at} - \alpha^2 t + t \right) \leq \\ &\leq \sup_{t>0} (1+t+\beta^2 t^2) \exp \left( -\frac{4a(\alpha^2 - 1) - \beta^2}{4a} t \right) < \infty. \end{aligned}$$

Case 3:  $x \geq \beta t$ .

$$\begin{aligned} \frac{(1+t+x^2) \exp(\lambda x - a\lambda^2 t)}{t + \exp(x^2 - t)} &\leq \\ &\leq (1+\beta^{-1}x+x^2) \exp(\lambda x - a\lambda^2 t - x^2 + t) \leq \\ &\leq \sup_x (1+\beta^{-1}x+x^2) \exp(\Lambda x - x^2 + \beta^{-1}x) < \infty. \end{aligned}$$

□

**3.4 Lemma.** For all  $a, \varepsilon, \Lambda \in (0, \infty)$ ,

$$\sup_{t>0, x \in \mathbb{R}, \lambda \in (-\Lambda, \Lambda)} \frac{(1+t+x^2)(1+\exp(\lambda x - a\lambda^2 t))}{t + \exp(\varepsilon x^2 - t)} < \infty.$$

*Proof.* By Lemma 3.3 applied to  $|x|, |\lambda|$ ,

$$\sup_{t>0, x \in \mathbb{R}, \lambda \in (-\Lambda, \Lambda)} \frac{(1+t+x^2) \exp(|\lambda x| - a\lambda^2 t)}{t + \exp(\varepsilon x^2 - t)} < \infty,$$

and  $\lambda x \leq |\lambda x|$ , of course. The new terms are bounded:

$$\begin{aligned} \frac{1+t}{t + \exp(\varepsilon x^2 - t)} &\leq \frac{1+t}{t + e^{-t}} \leq \frac{1+t}{\max(1, t)} \leq 2; \\ \frac{x^2}{t + \exp(\varepsilon x^2 - t)} &\leq \frac{x^2}{t + \varepsilon x^2 - t} \leq \frac{1}{\varepsilon}. \end{aligned}$$

□

Here is a counterpart of Lemma 2.6. This time, the origin  $\lambda = \eta = 0$  is included (but its neighborhood is reduced).

**3.5 Lemma.** For every  $a \in (0, \infty)$ , maps  $(\lambda, \eta) \mapsto \exp(\lambda X - \eta \tau)$  and  $(\lambda, \eta) \mapsto \tau \exp(\lambda X - \eta \tau)$  are continuous from  $\{(\lambda, \eta) : \lambda \in \mathbb{R}, \eta \in [a\lambda^2, \infty)\}$  to the space  $L_1$  of integrable random variables.

*Proof.* We apply the dominated convergence theorem, taking into account that  $\tau + \exp(\varepsilon X^2 - \tau)$  is an integrable majorant by Lemma 3.4. □

**3.6 Lemma.** For all  $a, \varepsilon, \Lambda \in (0, \infty)$ ,

$$\sup_{t>0, x \in \mathbb{R}, \lambda \in (-\Lambda, 0) \cup (0, \Lambda)} \frac{|\exp(\lambda x - a\lambda^2 t) - 1 - (\lambda x - a\lambda^2 t)|}{\lambda^2 (t + \exp(\varepsilon x^2 - t))} < \infty.$$

*Proof.* Denote  $u = \lambda x - a\lambda^2 t$ .

Case  $|x| \leq a|\lambda|t$ : we have  $|\lambda x| \leq a\lambda^2 t$ , thus  $-2a\lambda^2 t \leq u \leq 0$  and  $|e^u - 1 - u| = e^u - 1 - u \leq 1 - 1 - u = -u \leq 2a\lambda^2 t \leq 2a\lambda^2 (t + \exp(\varepsilon x^2 - t))$ .

Case  $|x| \geq a|\lambda|t$ : we apply the bound  $|e^u - 1 - u| \leq \frac{1}{2}u^2 \max(1, e^u)$ , note that  $u^2/\lambda^2 \leq 2x^2 + 2(a\lambda t)^2 \leq 4x^2$  and get an upper bound

$$\frac{x^2 \max(1, \exp(\lambda x - a\lambda^2 t))}{t + \exp(\varepsilon x^2 - t)},$$

bounded by Lemma 3.4. □

**3.7 Lemma.** For all  $a \in (0, \infty)$ ,

$$\frac{\mathbb{E} \exp(\lambda X - a\lambda^2 \tau) - 1}{\lambda^2} \rightarrow \frac{1}{2} - a \quad \text{as } \lambda \rightarrow 0.$$

*Proof.* We have

$$\frac{\exp(\lambda X - a\lambda^2\tau) - 1 - (\lambda X - a\lambda^2\tau)}{\lambda^2} \rightarrow \frac{1}{2}X^2 \quad \text{a.s. as } \lambda \rightarrow 0.$$

The left-hand side is dominated by  $\tau + \exp(\varepsilon X^2 - \tau)$  by Lemma 3.6, the majorant being integrable (for some  $\varepsilon$ ) by (1), (2). By the dominated convergence theorem,

$$\frac{\mathbb{E} \exp(\lambda X - a\lambda^2\tau) - 1 - \lambda \mathbb{E} X + a\lambda^2 \mathbb{E} \tau}{\lambda^2} \rightarrow \frac{1}{2} \mathbb{E} X^2;$$

it remains to use (3).  $\square$

Recall  $\eta_\lambda$  satisfying (5)  $\mathbb{E} \exp(\lambda X - \eta_\lambda \tau) = 1$ , given by Lemma 2.7;  $\eta_0 = 0$ , and  $\eta_\lambda > 0$  for  $\lambda \neq 0$ .

**3.8 Lemma.**  $\eta_\lambda = \frac{1}{2}\lambda^2 + o(\lambda^2)$  as  $\lambda \rightarrow 0$ .

*Proof.* If  $a > \frac{1}{2}$  then by Lemma 3.7,  $\mathbb{E} \exp(\lambda X - a\lambda^2\tau) < 1$  and therefore  $\eta_\lambda < a\lambda^2$  for all  $\lambda \neq 0$  small enough. Similarly, if  $a < \frac{1}{2}$  then  $\eta_\lambda > a\lambda^2$  for all  $\lambda \neq 0$  small enough.  $\square$

**3.9 Lemma.** The function  $\lambda \mapsto \mu_\lambda$  is continuous from  $\mathbb{R}$  to the space of measures with the norm topology.

*Proof.* Continuity on  $(-\infty, 0) \cup (0, \infty)$  holds by Lemma 2.8. The same proof gives now continuity at 0 due to Lemma 3.5 (and 3.8).  $\square$

**3.10 Lemma.** The set  $\{\mu_\lambda : \lambda \in [-C, C]\}$  satisfies the conditions of Theorem 1.3 whenever  $0 < C < \infty$ .

*Proof.* We repeat the proof of Lemma 2.9 using Lemma 3.9 instead of 2.8, and 3.5 instead of 2.6.  $\square$

Recall the functions  $h_\lambda(t) = e^{-\eta_\lambda t} \mu_\lambda((t, \infty))$ . Similarly to Lemma 2.10 we get uniform direct Riemann integrability of the set  $\{h_\lambda : \lambda \in [-C, C]\}$ . Similarly to Lemma 2.11, the map  $\lambda \mapsto h_\lambda$  is continuous from  $\mathbb{R}$  to  $L_1(0, \infty)$ . Similarly to Lemma 2.12, in the nonlattice case the map  $\lambda \mapsto (h_\lambda(k\delta))_{k=0}^\infty$  is continuous from  $\mathbb{R}$  to  $l_1$ .

*Proof of Theorem 3.* The first claim is given by Lemma 3.8. For the second claim, the proof of Theorem 2 needs only trivial modifications:  $[-C, C]$  and related results of Sect. 3 are used instead of  $[-C, -c] \cup [c, C]$  and related results of Sect. 2.  $\square$

*Proof of Theorem 1.* By Theorem 3,

$$\frac{1}{\lambda^2 t} \ln \mathbb{E} \exp \lambda S(t) = \frac{\eta_\lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2 t}\right) = \frac{1}{2} + o(1) + O\left(\frac{1}{\lambda^2 t}\right)$$

as  $t \rightarrow \infty$ , uniformly in  $\lambda \in [-C, 0) \cup (0, C]$ . Thus,

$$\lim_{t \rightarrow \infty, \lambda \rightarrow 0, \lambda^2 t \rightarrow \infty} \frac{1}{\lambda^2 t} \ln \mathbb{E} \exp \lambda S(t) = \frac{1}{2};$$

Theorem 1 follows by the well-known Gärtner(-Ellis) argument. □

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